

Renormalized Entanglement Entropy Flow in Mass-deformed ABJM Theory

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Abstract

We investigate a mass deformation effect on the renormalized entanglement entropy (REE) near the UV fixed point in (2+1)-dimensional field theory. In the context of the gauge/gravity duality, we use the Lin-Lunin-Maldacena (LLM) geometries corresponding to the vacua of the mass-deformed ABJM theory. We analytically compute the small mass effect for various droplet configurations and show in holographic point of view that the REE is monotonically decreasing, positive, and stationary at the UV fixed point. These properties of the REE in (2+1)-dimensions are consistent with the Zamolodchikov c -function proposed in (1+1)-dimensional conformal field theory.

Keywords : entanglement entropy, mass-deformed ABJM theory, LLM geometry
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1 Introduction

One of the well-known features of the entanglement entropy (EE) in quantum field theories is the appearance of the area law describing short range correlation. This correlation causes the UV divergence in the continuum limit and can be regulated in terms of the UV cutoff [1, 2]. In other words, the EE is UV sensitive. Due to this fact, it is not a good observable to measure the number of degrees of freedom related to the long range correlations of the ground state. For this reason, it is important to define a finite UV quantity in the continuum limit clearly, which plays a role of the Zamolodchikov c -function in 2-dimensional conformal field theory (CFT) [3].

Recently, Liu and Mezei proposed such a finite quantity defined from the EE in d -dimensional quantum field theory (QFT), so-called the renormalized entanglement entropy (REE) [4], where the size of entangling surface can be reinterpreted as the renormalization group (RG) flow scale. For the conformal fixed points in all dimensions, the REE with a spherical entangling surface reproduces the known central charge of a given CFT [5]. Except for the (1+1)-dimensional cases, however, understanding the REE along the RG flow still remains to be clarified.

The REE in (2+1)-dimensional QFT is defined as [4]

$$\mathcal{F}_{\text{disk}}(l) = \left(l \frac{\partial}{\partial l} - 1 \right) S_{\text{disk}}(l), \quad (1)$$

where $S_{\text{disk}}(l)$ denotes the EE for a disk with radius l . It was conjectured in [4] that the REE in 3-dimensions is an analogue of c -function which counts the number of degrees of freedom at a given energy scale, and then $\mathcal{F}_{\text{disk}}(l)$ satisfies the so called F -theorem [6, 7]. This conjecture for any (2+1)-dimensional Lorentz invariant field theories was proved in [8] using the strong subadditivity of the EE. See also [9, 10, 11] for related works.

There still remains an important issue on the stationarity of the REE at conformal fixed points. In contrast to the Zamolodchikov c -function, it was reported that $\mathcal{F}_{\text{disk}}(l)$ of a (2+1)-dimensional free massive field theory *would not be stationary* under a mass deformation [10]. This result is based on numerical evaluation of the EE for a disk [2, 12].

Related to the *stationarity* issue, we take a (2+1)-dimensional interacting CFT, the Aharony-Bergman-Jafferis-Maldacena (ABJM) theory [13] including mass deformation [14, 15]. In order to investigate the REE, we adopt the holographic method developed in [16, 17, 18]. It is known that the corresponding dual geometries are given by the Lin-Lunin-Maldacena (LLM) background with $\text{SO}(2,1) \times \text{SO}(4) \times \text{SO}(4)$ isometry in 11-dimensional supergravity [14, 15], which have one-to-one correspondence with the vacua of the mass-

deformed ABJM (mABJM) theory [19, 20]. These geometries have been actually conjectured to be dual to the supersymmetry preserving mass-deformation of the (2+1)-dimensional $\mathcal{N} = 8$ CFT even before the development of the ABJM theory at the Chern-Simons level $k = 1$ [15]. See also [21] for the related work.

In what follows, we review the relation between the vacua of the mABJM theory and the LLM geometry, and then compute $\mathcal{F}_{\text{disk}}(l)$ near the UV fixed point. We analytically compute the small mass effect of the REE for various droplet configurations in the LLM geometry and show the *stationarity* in (2+1)-dimensional strongly coupled supersymmetric massive field theory. Further elaboration for our results will be presented elsewhere [22].

2 Vacua of mABJM theory and dual LLM geometry

The $\mathcal{N} = 6$ mABJM theory has discrete vacua characterized by occupation numbers, N_n and \tilde{N}_n , of two types of the Gomis-Rodriguez-Gomez-Van Raamsdonk-Verlinde (GRVV) matrices [20]. See also [23, 24].

On the other hand, the LLM geometry under consideration is classified by droplet configurations for a given N M2-branes. Then N_n and \tilde{N}_n have one-to-one correspondence with the lengths of the black (l_n) and white (\tilde{l}_n) droplets in the LLM geometry. In [24], the explicit mapping is given to relate the vacua of the mABJM theory and the LLM geometry for general N and k . Therefore, if one picks up one special LLM geometry, one can immediately find a corresponding vacuum in the field theory side.

The LLM geometry with \mathbb{Z}_k quotient is given by

$$ds^2 = -e^{4\Phi/3} (-dt^2 + dw_1^2 + dw_2^2) + e^{-2\Phi/3} h^2 (dx^2 + dy^2) + e^{-2\Phi/3} y e^G ds_{\tilde{S}^3/\mathbb{Z}_k}^2 + e^{-2\Phi/3} y e^{-G} ds_{\tilde{S}^3/\mathbb{Z}_k}^2 \quad (2)$$

with

$$ds_{\tilde{S}^3/\mathbb{Z}_k}^2 = d\theta^2 + \sin^2 2\theta d\phi^2 + ((d\lambda + d\varphi/k) + \cos 2\theta d\phi)^2$$

$$ds_{\tilde{S}^3/\mathbb{Z}_k}^2 = d\tilde{\theta}^2 + \sin^2 2\tilde{\theta} d\tilde{\phi}^2 + ((-d\lambda + d\varphi/k) + \cos 2\tilde{\theta} d\tilde{\phi})^2,$$

where $e^{-2\Phi} = \mu_0^{-2}(h^2 - h^{-2}V^2)$, $h^{-2} = 2y \cosh G$, $z = \frac{1}{2} \tanh G$ with a mass parameter μ_0 given by the transverse 4-form field strength. Here, the parameters k and μ_0 are identified with the Chern-Simons level and the mass parameter, via $\mu_0 = \frac{\pi m}{2k}$, in the mABJM theory, respectively [24]. The LLM geometry (2) is completely determined in terms of $V(x, y)$ and $z(x, y)$,

$$z(x, y) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1} (x - x_i)}{2\sqrt{(x - x_i)^2 + y^2}},$$

$$V(x, y) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{2\sqrt{(x-x_i)^2 + y^2}}, \quad (3)$$

where x_i 's represent the locations of boundary lines between black and white strips in the droplet representation. See also [25] for the large N behavior of the LLM geometry.

3 Holographic EE for a Disk

The LLM geometry with \mathbb{Z}_k quotient becomes asymptotically $\text{AdS}_4 \times \text{S}^7 / \mathbb{Z}_k$. In the dual field theory side, this implies that the conformal symmetry is restored in the UV limit. Therefore, the asymptotic boundary of the LLM geometry is the same with that of the original ABJM theory. On the other hand, in the IR region, we see the breaking of conformal symmetry due to the mass deformation. This circumstance allows one to investigate the effect of the mass deformation near the UV conformal fixed point of the ABJM theory in the holographic point of view.

The purpose of this paper is to investigate the physical properties of REE near the fixed point of LLM geometry. We note that the value of REE at the UV fixed point is exactly known by the free energy of the ABJM theory [5]. The free energy itself was obtained in terms of the ABJM partition function on S^3 using the localization technique [26]. In order to obtain the REE near the UV fixed point of the mABJM theory, we first calculate the holographic EE (HEE) for a disk with radius l . The geometry we are considering is the (2+1)-dimensional flat Minkowski space with 7-dimensional compact space on the asymptotic boundary. We take a circular region with radius l in the two spatial directions of the Minkowski space. The expected 9-dimensional surface in HEE proposal is spanned by coordinates, σ^i , $i = 1, 2, \dots, 9$. Then the induced metric on the surface is given by

$$g_{ij} = \frac{\partial X^M \partial X^N}{\partial \sigma^i \partial \sigma^j} G_{MN}, \quad (4)$$

where $M, N = 0, \dots, 10$ and G_{MN} is the metric of the 11-dimensional LLM geometry.

The proposed HEE [16, 17] is given by

$$S_A = \frac{\text{Min}(\gamma_A)}{4G_N}, \quad \gamma_A = \int d^9 \sigma \sqrt{\det g_{ij}}, \quad (5)$$

where γ_A is the 9-dimensional surface represented by the induced metric (4), $\text{Min}(\gamma_A)$ the minimum value of γ_A , and $G_N = (2\pi l_P)^9 / (32\pi^2)$ the 11-dimensional Newton's constant with the Planck length l_P . To compute the HEE for a disk on the asymptotic boundary of the LLM geometry, we consider a mapping of coordinates in (2),

$$w_1 = \rho \cos \sigma^1, \quad w_2 = \rho \sin \sigma^1, \quad r = r(\rho), \quad \alpha = \sigma^3,$$

$$\theta = \sigma^4, \phi = \sigma^5, \tilde{\theta} = \sigma^6, \tilde{\phi} = \sigma^7, \lambda = \sigma^8, \varphi = \sigma^9, \quad (6)$$

where $r = \sqrt{x^2 + y^2}$, $\alpha = \tan^{-1}(y/x)$, $0 \leq \sigma^1 \leq 2\pi$, and $0 \leq \rho(= \sigma^2) \leq l$. Applying the mapping (6) to (5) and integrating out the coordinates of the two S^3 's and σ^1 , we obtain

$$\gamma_A = \frac{\pi^5 R^9}{16k\mu_0} \int_0^l d\rho \int_0^\pi d\alpha \frac{f\rho \sin^2 \alpha}{u^3} \sqrt{1 + \frac{f^2 u'^2}{4\mu_0^2 \sin^2 \alpha u^2}}, \quad (7)$$

where $u = \frac{R^3}{4r}$ with the radius of the AdS_4 , $R = (32\pi^2 kN)^{1/6} l_P$, and

$$f(u, \alpha) = \sqrt{1 - 4\tilde{z}^2 - 4\tilde{y}^2 \tilde{V}^2}. \quad (8)$$

Here we rescaled coordinates and functions as

$$\tilde{x} = \frac{4x}{R^2}, \quad \tilde{y} = \frac{4y}{R^2}, \quad \tilde{V}(\tilde{x}, \tilde{y}) = \frac{R^2}{4} V(x, y). \quad (9)$$

Under these rescalings, $z(x, y)$ does not change.

As shown in (7), the surface area γ_A depends on the function $f(u, \alpha)$ which includes all information about possible droplets of the LLM geometry. Therefore, in order to figure out the properties of the HEE for a given droplet configuration, one can analyze $f(u, \alpha)$ in various limits. In this paper, we are interested in the physics near the UV fixed point, and so it is enough to focus on the properties of γ_A for a small mass deformation.

The gauge/gravity duality implies that there is a correspondence between a strongly coupled gauge theory and a weakly curved gravity. As is well known, for the validity of this correspondence, one needs to take the large N limit. In our case, however, this large N limit is not enough to have well defined HEE: For a given N , the possible number of distinguishable LLM geometries is given by the partition of N , $p(N)$, and it behaves as $p(N) \sim e^{\pi\sqrt{\frac{2N}{3}}}$ in the large N limit. It is not sure that the geometries for all such possibilities are weakly curved over the whole space-time region. Thus it is necessary to take a step of selecting suitable geometries which are weakly curved in the large N limit. Fortunately, there is a guideline to select weakly curved geometries. In the Young diagram representation of the LLM geometry, the lengths of the white/black strips are mapped to those of the horizontal/vertical edges of the Young diagram. The area of the diagram is identified with N . For geometries without strongly curved region, one has to consider the Young diagram including only the long edges which are of the order of \sqrt{N} [15]. If some of the edges are short ($\ll \sqrt{N}$), then their presence makes the geometry highly curved. This characteristic feature of the LLM geometries has been observed in [25]. From the curvature behavior of the LLM geometry, we conclude that the geometry corresponding to

the rectangular shaped Young diagram with the side length of order \sqrt{N} is weakly curved over the whole space-time region [25]. In this case the gauge/gravity duality is well-defined. From now on, we will concentrate on the LLM geometry having such rectangular shaped Young diagram.

Under the small mass deformation, we expand $f(u, \alpha)$ in (8) for a rectangular shaped Young diagram with side lengths, denoted by w and b , as

$$f(u, \alpha) = 2 \sin \alpha (\mu_0 u) \left[1 + \frac{\tilde{\sigma} \cos \alpha}{\sqrt{2}} (\mu_0 u) + \frac{\tilde{\sigma}^2 - 1 + (5\tilde{\sigma}^2 + 9) \cos 2\alpha}{8} (\mu_0 u)^2 + \mathcal{O}((\mu_0 u)^3) \right], \quad (10)$$

where $w = \frac{\sqrt{kN}}{\hat{\sigma}}$, $b = \hat{\sigma} \sqrt{kN}$, and $\tilde{\sigma} = \hat{\sigma} - \frac{1}{\hat{\sigma}}$. For the fully symmetric case, $\hat{\sigma} = 1$ and hence $\tilde{\sigma} = 0$. As discussed in the previous paragraph, $f(u, \alpha)$ is valid for $|\tilde{\sigma}| \ll \sqrt{kN}$.

After the integration over α , the surface area γ_A is expanded in terms of small mass as follows:

$$\gamma_A = \frac{\pi^5 R^9}{6k} \int_0^l d\rho \rho \left[\frac{1}{u^2} \sqrt{1 + u'^2} - \frac{5\tilde{\sigma}^2 + 16}{20(1 + u'^2)^{3/2}} \mu_0^2 - \frac{3(\tilde{\sigma}^2 + 4)(3u'^2 + 2u'^4)}{5(1 + u'^2)^{3/2}} \mu_0^2 + \dots \right]. \quad (11)$$

The minimum value of γ_A is given by the solution of the equation of motion for $u(\rho)$ after regarding γ_A as a classical Euclidean action. In the $\mu_0 \rightarrow 0$ limit, γ_A is reduced to the case of the AdS_4 up to an overall factor coming from the contribution of the 7-dimensional compact space. In this limit, a special solution is known, satisfying the appropriate boundary conditions, $u'_0(0) = 0$ and $u_0(l) = 0$ [17],

$$u_0(\rho) = \sqrt{l^2 - \rho^2}. \quad (12)$$

In order to see the effect of mass deformation near the UV fixed point, we consider the perturbation with the mass parameter μ_0 around the solution $u_0(\rho)$. Since the first correction appears at μ_0^2 order and we are interested only in the leading order correction to γ_A , we take

$$u(\rho) = u_0(\rho) + (\mu_0 l)^2 \delta u(\rho). \quad (13)$$

From the equation of motion for $u(\rho)$, it is possible to get a general solution of $\delta u(\rho)$. By imposing two boundary conditions, $\delta u'(0) = 0$ and $\delta u(l) = 0$, two integration constants are fixed and what we obtain at the end is

$$\delta u(\rho) = \frac{l^3}{300 \sqrt{1 - (\rho/l)^2}} \left[206\tilde{\sigma}^2 + 800 \right]$$

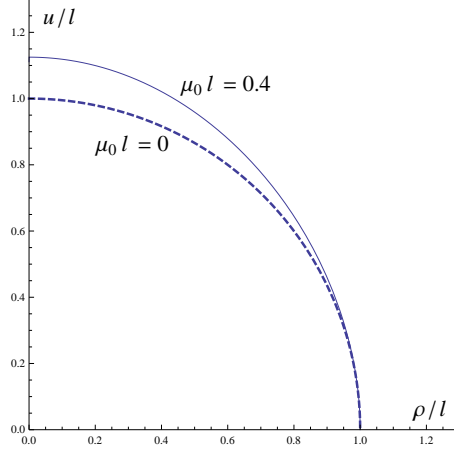


Figure 1: Change of the minimal surface due to the mass deformation in the symmetric case $\tilde{\sigma} = 0$

$$\begin{aligned}
& \times \left(\tanh^{-1} \sqrt{1 - (\rho/l)^2} + \ln(\rho/l) - \sqrt{1 - (\rho/l)^2} \right) \\
& + \frac{81\tilde{\sigma}^2}{2} (\rho/l)^6 + \frac{160 - 11\tilde{\sigma}^2}{2} (\rho/l)^4 \\
& - (560 + 119\tilde{\sigma}^2)(\rho/l)^2 + \frac{1120 + 281\tilde{\sigma}^2}{2} \Big].
\end{aligned} \tag{14}$$

The effect of mass deformation on the minimal surface is depicted in Fig. 1.

Plugging (13) into (11) and performing the ρ integration, we obtain the HEE from (5) for a disk with radius l , up to μ_0^2 order,

$$S_{\text{disk}} = \frac{\pi^5 R^9}{12kG_N} \left(\frac{l}{\epsilon} - 1 - \frac{103\tilde{\sigma}^2 + 400}{300} (l\mu_0)^2 \right), \tag{15}$$

where the constant ϵ is the UV cutoff in the u coordinate. The first two terms correspond to the HEE of the ABJM theory without mass deformation. The last term which is one of our main results is the leading correction from the small mass expansion. If the idea of HEE is correct, the above expression is interpreted as the EE for the strongly interacting massive field theory. Although it is hard to compute the EE on the field theory side, it would be desirable to check whether the HEE of (15) matches with the would-be field theory result.

4 REE and c -function in (2+1)-dimensions

From the HEE in (15) for the rectangular droplets, one obtains the REE defined in (1), up to μ_0^2 order,

$$\mathcal{F}_{\text{disk}}(l) = F - \frac{\pi^5 R^9 (103\tilde{\sigma}^2 + 400)}{3600kG_N} (l\mu_0)^2, \tag{16}$$

where $F = \frac{\pi^5 R^9}{12kG_N}$ is the free energy of the ABJM theory without mass deformation. As we explained previously, the REE can be a c -function in holographic point of view, which counts the number of effective degrees of freedom of a given system at the length scale l . The REE in (16) shows that the expected holographic c -function near the UV fixed point is positive and monotonically decreases along the RG flow as the system size l increases. This result supports the F -theorem in 3-dimensional field theory [6]. The REE in (16) describes the behavior of the c -function around the UV fixed point.

The conformal symmetry of the ABJM theory is broken by perturbing in terms of the fermionic mass term [19, 20, 27, 28],

$$\mathcal{L}_{\text{ferm}} \sim \mu_0 \Psi^{\dagger A} M_A^B \Psi_B \quad (17)$$

with the mass matrix $M_A^B = \text{diag}(1, 1, -1, -1)$. Then the dimensionless coupling constant for the relevant mass deformation is $g = l\mu_0$, and the corresponding coupling constant for the bosonic scalar fields is proportional to g^2 due to the supersymmetry. As for the stationarity issue at the UV fixed point in (2+1)-dimensional interacting field theory [10], we note that the REE of (16) leads to¹

$$\left. \frac{\partial \mathcal{F}_{\text{disk}}}{\partial g} \right|_{g=0} = 0. \quad (18)$$

This result shows that the REE in (2+1)-dimensional CFT is stationary under the relevant mass deformation, like the c -function in (1+1)-dimensions [3] and the a -function in (3+1)-dimensions [29].

5 Summary

In this paper we have investigated the UV behavior of the REE for a disk in (2+1)-dimensional field theory. In holographic point of view we have confirmed analytically that the REE is positive, monotonically decreasing along the RG flow, and stationary under a relevant mass deformation at the UV conformal fixed point. As a top-down approach for the gauge/gravity duality, we have contemplated on the 11-dimensional LLM geometries dual to the vacua of the mABJM theory. Our analytic results can be applied to various droplet configurations in the LLM geometry.

Near the UV fixed point, the roles of the REE are the same with the Zamolodchikov c -function in 2-dimensions and the a -function in 4-dimensions. We would like to emphasize

¹This result is the check for a supersymmetric mass deformation using analytic calculations, but it is not clear whether non-supersymmetric mass deformations [10] guarantees or not the stationarity in strongly coupled models. We need more investigations in this direction.

that we have considered an interacting theory and given an analytic result. Our computations show that the REE, which counts the effective degrees of freedom at a given length (energy) scale, can be a strong candidate for the c -function in 3-dimensions.

As a final remark, the investigation of the IR region for the mABJM theory will certainly give some insights on other aspects of the REE.

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